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VISCOSITY APPROXIMATION METHODS FOR A COMMON FIXED POINT OF A FINITE FAMILY OF GENERALIZED ASYMPTOTICALLY QUASI-NONEXPANSIVE MAPPINGS IN BANACH SPACES

JAMNIAN NANTADILOK

Department of Mathematics
Faculty of Science
Lampang Rajabhat University
Lampang, 52100
Thailand
e-mail: jamnian52@lpru.ac.th

Abstract

Let K be a nonempty closed convex subset of a real reflexive Banach spaces E that has weakly continuous duality mapping J_φ, for some guage φ. Let $T_i : K \to K, i = 1, 2, \dots, N$ be a finite family of generalized asymptotically quasi-nonexpansive mappings with $F := \bigcap_{i\geq 1}^{N} F(T_i) \neq \emptyset$, which is a sunny nonexpansive retract of K with Q, a nonexpansive retraction. For $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm $x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) T_j^n (x_n)$, $j \geq 0, n \geq 1$, where $f : K \to K$ is a contraction mapping, and let $\{\alpha_n\} \subseteq (0, 1)$ be a sequence satisfying certain conditions. Suppose that $\{x_n\}$ satisfies condition (A).

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Then, it is proved that $\{x_n\}$ converges strongly to a common fixed point $\bar{x} = Qf(\bar{x})$ of a finite family T_i, $i = 1, 2, \ldots, N$. Moreover, $\bar{x}$ is the unique solution in F to a certain variational inequaliy.

1. Introduction

Let E be a real Banach space with dual E^*. A guage function is a continuous strictly increasing function $\varphi : R^+ \to R^+$, such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$. The duality mapping $J_\varphi : E \to E^*$ associated with a guage function φ is defined by $J_\varphi := \{u^* : \langle x, u^* \rangle = \|x\| \cdot \|u^*\|, \|u^*\| = \varphi(\|x\|)\}$, $x \in E$, where $\langle .,. \rangle$ denotes the generalized duality pairing. In the particular case $\varphi(t) = t$, the duality map $J = J_\varphi$ is called the *normalized duality map*. We denote that $J_\varphi = \frac{\varphi(\|x\|)}{\|x\|} J(x)$. It is known that, if E is smooth, then J_φ is single valued and norm to w^* continuous (see, e.g., [5]).

Following Browder [2], we say that a Banach space E has *weakly continuous duality mapping*, if there exists a guage function φ, for which the duality map J_φ is single valued and weak to weak* sequentially continuous (i.e., if $\{x_n\}$ is a sequence in E weakly convergent to a point x, then the sequence $\{J_\varphi(x_n)\}$ converges weak* to $J_\varphi(x)$).

It is known that $l^p (1 < p < \infty)$ spaces have a weakly continuous duality mapping J_φ with a guage $\varphi(t) = t^{p-1}$.

Setting

$$\Phi(t) = \int_0^t \varphi(\tau) d\tau, \ t \geq 0, \tag{1.1}$$

one can see that $\Phi(t)$ is a convex function and $J_\varphi(x) = \partial\Phi(\|x\|)$, for $x \in E$, where ∂ denotes the subdifferential in the sense of convex analysis.

Let K be a nonempty closed convex subset of a real Banach space E. A mapping $T : K \to E$ is called *generalized asymptotically quasi-nonexpansive* with respect to sequence $\{r_n\}$ and $\{s_n\} \subset [0, 1)$ with $r_n \to 0$ and $s_n \to 0$ as $n \to \infty$, if $\forall x \in K$ and $p \in F(T)$, the following inequality holds:

$$\|T^n x - p\| \leq (1 + r_n)\|x - p\| + s_n\|x - T^n x\|, \quad n \geq 1, \tag{1.2}$$

where $F(T) := \{x \in K : T(x) = x\} \neq \emptyset$. T is called *quasi-nonexpansive*, if $\|Tx - p\| \leq \|x - p\|$, $\forall x \in K$, and $p \in F(T)$, and T is called *nonexpansive*, if $\|Tx - y\| \leq \|x - y\|$, $\forall x, y \in K$. It is clear that a nonexpansive mapping T with $F(T) \neq \emptyset$ is quasi-nonexpansive, and it is known that every quasi-nonexpansive mapping is generalized asymptotically quasi-nonexpansive (see, e.g., [7]). The converse is not true. For a sequence $\{\alpha_n\} \in (0, 1)$ and an arbitrary $u \in K$, let the sequence $\{x_n\} \in K$ be iteratively defined by $x_0 \in K$,

$$x_{n+1} := \alpha_{n+1}u + (1 - \alpha_{n+1})T(x_n), \quad n \geq 0, \tag{1.3}$$

where T is a nonexpansive mapping of K into itself.

Hapern [6] was the first to study the convergence of the algorithm (1.3) in the framework of Hilbert spaces. Lions [9] improved the result of Hapern, still in Hilbert spaces, by proving strong convergence of $\{x_n\}$ to a fixed point of T, if the real sequence $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0; \quad \text{(ii)} \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \text{and (iii)} \lim_{n\to\infty} \frac{\alpha_n - \alpha_{n-1}}{\alpha_n^2} = 0. \tag{1.4}$$

Wittmann [12] proved, still in Hilbert spaces, the strong convergence of $\{x_n\}$, if $\{\alpha_n\}$ satisfies the following conditions:

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0; \quad \text{(ii)} \sum_{n=1}^{\infty} \alpha_n = \infty; \quad \text{and (iii)} \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty. \tag{1.5}$$

Reich [11] extended the result of Wittmann to the class of Banach spaces, which are uniformly smooth and have weakly sequentially continuous duality mappings.

In 2000, Moudafi [10] introduced viscosity approximation method and proved that, if E is a real Hilbert space, for given $x_0 \in K$, the sequence $\{x_n\}$ generated by the algorithm:

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n)T(x_n), \quad n \geq 0, \tag{1.6}$$

where $f : K \to K$ is a contraction mapping with constant $\beta \in (0, 1)$, and $\{\alpha_n\} \subseteq (0, 1)$ satisfies certain conditions, converges strongly to a fixed point of T in K, which is the unique solution to the following variational inequality:

$$\langle (I - f)x^*, x - x^* \rangle \geq 0, \forall x \in F(T).$$

Moudafi in [10], generalizes Browder's and Hapern's theorems in the direction of viscosity approximations. Viscosity approximations are very important because they are applied to convex optimization, linear programming, monotone inclusions, and elliptic differential equations.

In 2004, Xu [14] studied further the viscosity approximation method for non-expansive mappings in uniformly smooth Banach spaces. This result of Xu [14] extends Theorem 2.2 of Moudfi [10] to Banach space setting. For details on the iterative methods, we refer the reader to [1].

Our concern now is the following. *Is it possible to construct a viscosity approximation sequence*, *which converges to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings in Banach spaces*?

Let K be closed and convex subset of a Banach space E. Let $T_i : K \to K$, $i = 1, 2, \ldots, N$, be a finite family of generalized asymptotically quasi-nonexpansive mappings with $F := \bigcap_{i \geq 1}^{N} F(T_i) \neq \emptyset$, which is a sunny nonexpansive retract of K. Let $f : K \to K$ be a contraction. We consider the problem of finding

$$\bar{x} \in K \text{ such that } \bar{x} = Q(f(\bar{x})), \tag{1.7}$$

where $Q : K \to F$ is a sunny nonexpansive retraction. We observe that $\bar{x}$ is a unique solution. Indeed, since F is a sunny nonexpansive retract of K, we have that $Q(f)$ is contraction. It follows from the standard Banach contraction principle that the point $\bar{x}$ is unique in (1.7).

For $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) T_j^n(x_n), \quad j \geq 0,\ n \geq 1, \tag{1.8}$$

where $\{\alpha_n\} \subseteq (0, 1)$ satisfies the following conditions:

$$\text{(i)} \lim_{n\to\infty} \alpha_n = 0; \quad \text{(ii)} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ is said to satisfy *condition* (A), if for any subsequence $x_{n_k} \rightharpoonup x$, and $x_{n+1} - T_n(x_n) \to 0$ implies $x \in F$.

It is our purpose in this paper to prove the convergence of viscosity approximation scheme (1.8) to a common fixed point of a finite family of generalized asymptotically quasi-nonexpansive mappings more general class of mappings in Banach spaces. This provides an affirmative answer to the above question.

2. Preliminaries

Let $K \subseteq E$ be closed convex and Q be a mapping of E onto K. Then Q is said to be *sunny*, if $Q(Q(x) + t(x - Q(x))) = Q(x)$, for all $x \in E$ and $t \geq 0$. A mapping Q of E into E is said to be *retraction*, if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$, where $R(Q)$ is the range of Q. A subset K of E is said to be a *sunny nonexpansive retract* of E, if there exists a sunny nonexpansive retraction of E onto K, and it is said to be a nonexpansive retract of E, if there exists a nonexpansive retraction of E onto K. If $E = H$, *the metric projection* P_K *is a sunny nonexpansive retraction from* H *to any closed convex subset of* H.

In what follows, we shall make use of the following lemmas.

Lemma 2.1 (See, e.g., [3]). *Let E be a smooth Banach space and K be a nonempty subset of E. Let $Q : E \to K$ be a retraction and J be the normalized duality map on E. Then, the following are equivalent*:

(i) *Q is sunny nonexpansive*;

(ii) $\langle x - Q(x), J(y - Q(x))\rangle \leq 0$ *for all* $x \in E$ *and* $y \in K$.

We note that Lemma 2.1 still holds, if the normalized duality map J is replaced with the general duality map J_{φ}, where φ is a guage function.

Lemma 2.2 (See, e.g., [7]). *Let $\{a_n\}, \{b_n\}$, and $\{\delta_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty}b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.3 (See, e.g., [13]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n\mu_n, \quad n \geq 0,$$

where (i) $0 < \alpha_n < 1$; (ii) $\sum_{n=1}^{\infty}\alpha_n = \infty$. *Suppose, either* (a) $\sigma_n = o(\alpha_n)$, *or* (b) $\sum_{n=1}^{\infty}\sigma_n < \infty$, *where* $\sigma_n = \alpha_n\mu_n$, *or* (c) $\limsup_{n\to\infty} \mu_n \leq 0$. *Then* $a_n \to 0$ *as* $n \to \infty$.

Let Φ be defined as in (1.1). Then, the next lemma is an immediate consequence of the subdifferential inequality.

Lemma 2.4 (See, e.g., [5]). *Let E be a real Banach space. Then for all $x, y \in E$, we get that*

$$\Phi(\|x + y\|) \leq \Phi(\|x\|) + \langle y, j_{\varphi}(x + y)\rangle, \quad \forall j_{\varphi} \in J_{\varphi}. \tag{2.1}$$

3. Main Results

Theorem 3.1. *Let K be a nonempty closed convex subset of a real reflexive Banach space E, that has weakly continuous duality mapping J_φ for some guage φ. Let $T_i : K \to K$, $i = 1, 2, \ldots, N$, be a finite family of generalized asymptotically quasi-nonexpansive mappings with respect to $\{r_{in}\}$ and $\{s_{in}\}$ such that $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, and*

$$F = Fix(T_N \ldots T_1) = Fix(T_1 T_N \ldots T_3 T_2) = \cdots = Fix(T_{N-1} T_{N-2} \ldots T_1 T_N),$$

and F is sunny nonexpansive retract of K with Q, a nonexpansive retraction, and $\sum_{n=1}^{\infty} \frac{r_n + 2s_n}{1 - s_n} < \infty$, where $r_n = \max\{r_{in} : i = 1, 2, \ldots, N\}$ and $s_n = \max\{s_{in} : i = 1, 2, \ldots, N\}$. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) T_j^n(x_n), \quad j \geq 0, \; n \geq 1, \tag{3.1}$$

where $T_n = T_{n \bmod N}$, where $f : K \to K$ is a contraction mapping with constant $\beta \in (0, 1)$ and $\{\alpha_n\} \subseteq (0, 1)$ satisfies the following conditions:

$$\text{(i)} \lim_{n \to \infty} \alpha_n = 0; \quad \text{(ii)} \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Suppose that $\{x_n\}$ satisfies condition (A). *Then, $\{x_n\}$ converges strongly to a common fixed point $\bar{x} = Q(f(\bar{x}))$ of a finite family T_i, $i = 1, 2, \ldots, N$, as $n \to \infty$. Moreover, $\bar{x}$ is the unique solution in F to the variational inequality*

$$\langle f(\bar{x}) - \bar{x}, \, j_\varphi(y - \bar{x}) \rangle \leq 0, \; \forall y \in F. \tag{3.2}$$

Proof. Let $p \in \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$. Then from (1.2), we obtain that

$$\|x_n - T_j^n x_n\| \leq \|x_n - p\| + \|T_j^n x_n - p\|$$

$$\leq \|x_n - p\| + (1 + r_{jn})\|x_n - p\| + s_{jn}\|x_n - T_j^n x_n\|$$

$$\leq (2 + r_{jn})\|x_n - p\| + s_{jn}\|x_n - T_j^n x_n\|$$

$$\leq (2 + r_n)\|x_n - p\| + s_n\|x_n - T_j^n x_n\|,$$

which implies that

$$\|x_n - T_j^n x_n\| \leq \frac{2 + r_n}{1 - s_n}\|x_n - p\|. \tag{3.3}$$

Therefore, it follows from (1.2), (3.1), and (3.3) that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|x_{n+1} - \alpha_n f(p) - (1 - \alpha_n)p + \alpha_n f(p) + (1 - \alpha_n)p - p\| \\
&\leq \|x_{n+1} - \alpha_n f(p) - (1 - \alpha_n)p\| + \|\alpha_n f(p) + (1 - \alpha_n)p - p\| \\
&\leq \|\alpha_n f(x_n) + (1 - \alpha_n)T_j^n(x_n) - \alpha_n f(p) - (1 - \alpha_n)p\| + \alpha_n\|f(p) - p\| \\
&= \|\alpha_n(f(x_n) - f(p)) + (1 - \alpha_n)(T_j^n(x_n) - p)\| + \alpha_n\|f(p) - p\| \\
&\leq \alpha_n\beta\|x_n - p\| + (1 - \alpha_n)\left[(1 + r_{jn})\|x_n - p\| + s_{jn}\|x_n - T_j^n(x_n)\|\right] \\
&\quad + \alpha_n\|f(p) - p\| \\
&\leq \left(\alpha_n\beta\|x_n - p\| + (1 - \alpha_n)(1 + r_n)\|x_n - p\|\right) \\
&\quad + (1 - \alpha_n)s_n\|x_n - T_j^n(x_n)\| + \alpha_n\|f(p) - p\| \\
&\leq \left[\alpha_n\beta + (1 - \alpha_n)(1 + r_n)\right]\|x_n - p\| + (1 - \alpha_n)s_n\frac{2 + r_n}{1 - s_n}\|x_n - p\| \\
&\quad + \alpha_n\|f(p) - p\| \\
&= \left[\alpha_n\beta + (1 - \alpha_n)\left(\frac{1 + r_n + s_n}{1 - s_n}\right)\right]\|x_n - p\| + \alpha_n\|f(p) - p\| \\
&= \left[\alpha_n\beta + (1 - \alpha_n)k_n\right]\|x_n - p\| + \alpha_n\|f(p) - p\|.
\end{aligned}$$

We can write the above inequality as follows

$$\|x_{n+1} - p\| \le (1 + \delta_n)\|x_n - p\| + b_n, \tag{3.4}$$

where $\delta_n = (\alpha_n\beta + (1 - \alpha_n)k_n) - 1$, and $b_n = \alpha_n\|f(p) - p\|$, and $k_n = \frac{1 + r_n + s_n}{1 - s_n}$. Since $k_n \to 1$ as $n \to \infty$, we can see that

$$\sum_{n=1}^{\infty}\delta_n < \infty, \quad \sum_{n=1}^{\infty}b_n < \infty. \tag{3.5}$$

It follows from Lemma 2.2 that $\lim_{n\to\infty}\|x_n - p\|$ exists, and hence the sequence $\{x_n\}$ is bounded. Furthermore, since $F \neq \emptyset$, we get that $\{f(x_n)\}$ and $\{T_j^n(x_n)\}$ are bounded. But this implies that

$$\|x_{n+1} - T_n^n(x_n)\| = \alpha_n\|f(x_n) - T_n^n(x_n)\| \to 0, \text{ as } n \to \infty.$$

In particular, we have that

$$\|x_{n+1} - T_n(x_n)\| \to 0, \text{ as } n \to \infty. \tag{3.6}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x},\ j_\varphi(x_{n+1} - \bar{x}) \le 0.$$

Since E is reflexive and $\{x_n\}$ is bounded, we may assume $x_{n_k} \rightharpoonup \omega$ such that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x},\ j_\varphi(x_{n+1} - \bar{x})\rangle = \lim_{k\to\infty}\langle f(\bar{x}) - \bar{x},\ j_\varphi(x_{n_k} - \bar{x})\rangle.$$

Then from (3.6) and our assumption, we obtain that $\omega \in F$. On the other hand, from the standard characterization of retraction onto F and the assumption that, the duality mapping J_φ is weakly continuous, Lemma 2.1 gives that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x},\ j_\varphi(x_{n+1} - \bar{x})\rangle &= \lim_{k\to\infty}\langle f(\bar{x}) - \bar{x},\ j_\varphi(x_{n_k} - \bar{x})\rangle \\ &= \langle f(\bar{x}) - \bar{x},\ j_\varphi(\omega - \bar{x})\rangle \\ &\le 0. \end{aligned} \tag{3.7}$$

Finally, we show that $x_n \to \bar{x}$ as $n \to \infty$. From (3.1) and Lemma 2.4, we get that

$$\begin{aligned}
\Phi\left(\|x_{n+1} - \bar{x}\|\right) &= \Phi\Big(\|\alpha_n(f(x_n) - f(\bar{x})) + (1 - \alpha_n)(T_i^n(x_n) - \bar{x})\| \\
&\quad + \alpha_n(f(\bar{x}) - \bar{x})\Big) \\
&\le \Phi\left(\|\alpha_n(f(x_n) - f(\bar{x})) + (1 - \alpha_n)(T_i^n(x_n) - \bar{x})\|\right) \\
&\quad + \alpha_n\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x})\rangle \\
&\le \Phi\Big(\alpha_n\beta\|x_n - \bar{x}\| + (1 - \alpha_n)[(1 + r_n)\|x_n - \bar{x}\| \\
&\quad + s_n\|x_n - T_i^n(x_n)\|]\Big) + \alpha_n\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x})\rangle \\
&\le \Phi\Big(\alpha_n\beta\|x_n - \bar{x}\| + (1 - \alpha_n)(1 + r_n)\|x_n - \bar{x}\| \\
&\quad + (1 - \alpha_n)s_n\frac{2 + r_n}{1 - s_n}\|x_n - \bar{x}\|\Big) + \alpha_n\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x})\rangle \\
&\le \Phi\left(\left[\alpha_n\beta + (1 - \alpha_n)k_n\right]\|x_n - \bar{x}\|\right) + \alpha_n\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x})\rangle \\
&\le \left[\alpha_n\beta + (1 - \alpha_n)k_n\right]\Phi\left(\|x_n - \bar{x}\|\right) + \alpha_n\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x})\rangle.
\end{aligned}$$

We can write the above inequality as follows

$$\Phi\left(\|x_{n+1} - \bar{x}\|\right) \le (1 - a_n)\Phi\left(\|x_n - \bar{x}\|\right) + a_n\left(\frac{\alpha_n}{a_n}\right)\langle f(\bar{x}) - \bar{x}, j_\varphi(x_{n+1} - \bar{x})\rangle,$$

where $a_n = 1 - [\alpha_n\beta + (1 - \alpha_n)]k_n$ and $k_n = \dfrac{1 + r_n + s_n}{1 - s_n} \to 1$ as $n \to \infty$. We can see that (i) $0 < a_n < 1$ and (ii) $\sum_{n=1}^{\infty} a_n = \infty$. Thus, (3.7) and Lemma 2.3 give that $x_n \to \bar{x}$ as $n \to \infty$. Moreover, $\bar{x}$ satisfying condition (3.2) follows from the property of Q. To show it is unique, let

$\overline{y} \in F$ be another solution of (3.2) in F. Then adding $\langle f(\overline{x}) - \overline{x}, j_\varphi(\overline{y} - \overline{x}) \rangle \le 0$ and $\langle f(\overline{y}) - \overline{y}, j_\varphi(\overline{x} - \overline{y}) \rangle \le 0$, we get that $(1 - \beta)(\|\overline{x} - \overline{y}\|) \|\overline{x} - \overline{y}\| \le 0$. This gives $\overline{x} = \overline{y}$. This completes the proof.

□

Remark 3.2. If, in Theorem 3.1, K is a compact subset of a real smooth Banach space E, then the weak continuity of J_φ may not be needed. In fact, we have the following corollary.

Corollary 3.3. *Let K be a nonempty convex and compact subset of a real smooth Banach space E. Let $T_i : K \to K$, $i = 1, 2, \ldots, N$, be a finite family of generalized asymptotically quasi-nonexpansive mappings with $F := \bigcap_{i=1}^{N} F(T_i) \neq \emptyset$, which is a sunny nonexpansive retract of K with Q, a nonexpansive retraction. For given $x_0 \in K$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} := \alpha_n f(x_n) + (1 - \alpha_n) T_j^n(x_n), \quad j \ge 0, \; n \ge 1, \tag{3.8}$$

where $T_n = T_{n \bmod N}$, and $f : K \to K$ is a contraction mapping with constant $\beta \in (0, 1)$ satisfies the following conditions: (i) $\lim_{n\to\infty} \alpha_n = 0$; (ii) $\sum_{n=1}^{\infty} \alpha_n = \infty$. *Suppose that $\{x_n\}$ satisfies condition* (A). *Then $\{x_n\}$ converges strongly to a common fixed point $\overline{x} = Q(f(\overline{x}))$ of T_i, $i = 1, 2, \ldots, N$, as $n \to \infty$. Moreover, $\overline{x}$ is the unique solution in F to the following variational inequality*:

$$\langle f(\overline{x}) - \overline{x}, j_\varphi(y - \overline{x}) \rangle \le 0, \; \forall y \in F. \tag{3.9}$$

Proof. Following the method of proof of Theorem 3.1, we get that

$$\|x_{n+1} - T_n(x_n)\| \to 0 \text{ as } n \to \infty. \tag{3.10}$$

Next, we show that

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x},\ j_{\varphi}(x_{n+1} - \bar{x})\rangle \leq 0.$$

Since K is compact and $\{x_n\}$ is bounded, we can assume that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \omega \in K$,

$$\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x},\ j_{\varphi}(x_{n+1} - \bar{x})\rangle = \lim_{k\to\infty}\langle f(\bar{x}) - \bar{x},\ j_{\varphi}(x_{n_k} - \bar{x})\rangle.$$

Then from (3.10) and our assumption, we obtain that $\omega \in F$. On the other hand, from the fact that E is smooth, the duality mapping J_{φ} being norm to weak* continuous and the standard characterization of retraction onto F, we obtain that

$$\begin{aligned}\limsup_{n\to\infty}\langle f(\bar{x}) - \bar{x},\ j_{\varphi}(x_{n+1} - \bar{x})\rangle &= \lim_{k\to\infty}\langle f(\bar{x}) - \bar{x},\ j_{\varphi}(x_{n_k} - \bar{x})\rangle \\ &= \langle f(\bar{x}) - \bar{x},\ j_{\varphi}(\omega - \bar{x})\rangle \\ &\leq 0.\end{aligned}$$

Now, following the proof of Theorem 3.1, we get the required result. This completes the proof. □

Remark 3.4. Our results extend and improve the corresponding results of Moudafi [10], and extend the corresponding result of Xu [14], Chang [4] to a more general class of nonexpansive mappings.

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